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Time-Space Noncommutativity: Quantised Evolutions

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Abstract

In previous work [7], we developed quantum physics on the Moyal plane with time-space noncommutativity, basing ourselves on the work of Doplicher et al. [1]. Here we extend it to certain noncommutative versions of the cylinder, $\mathbb{R}^3$ and $\mathbb{R} \times S^3$. In all these models, only discrete time translations are possible, a result known before in the first two cases [2]-[6]. One striking consequence of quantised time translations is that even though a time independent Hamiltonian is an observable, in scattering processes, it is conserved only modulo $\frac{2\pi}{\theta}$, where θ is the noncommutative parameter. (In contrast, on a one-dimensional periodic lattice of lattice spacing a and length $L = Na$, only momentum mod $\frac{2\pi}{L}$ is observable (and can be conserved).) Suggestions for further study of this effect are made. Scattering theory is formulated and an approach to quantum field theory is outlined.

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I. Introduction

Let $x = (x_0\,,\,\vec{x}) \in \mathbb{R}^d$ be coordinates of $\mathbb{R}^d$ with x_0 and $x_i\,(i = 1, 2, 3)$ being its time and spatial components. The coordinate functions $\hat{x}_\mu$ are then defined by evaluation map:

$$\hat{x}_\mu\,(x) = x_\mu\,. \tag{1}$$

The algebra generated by $\hat{x}_\mu$ with $*$-operation $\hat{x}_\mu^* = \hat{x}_\mu$ is the C^*-algebra $\mathcal{C}^0\,(\mathbb{R}^d) := \mathcal{A}_0(\mathbb{R}^d)$ of $\mathbb{R}^d$.

The Moyal plane $\mathcal{A}_\theta(\mathbb{R}^d)$ is a deformation of this algebra where $\hat{x}_\mu$ do not commute:

$$[\,\hat{x}_\mu\,,\,\hat{x}_\nu\,] = i\,\theta_{\mu\nu}\mathbb{I}\,,$$

$$\theta_{\mu\nu} = \text{real constants}, \quad \theta_{\mu\nu} = -\theta_{\nu\mu}\,. \tag{2}$$

In previous work [7], we developed quantum mechanics on $\mathcal{A}_\theta(\mathbb{R}^d)$ with time-space noncommutativity,

$$\theta_{0i} \neq 0\,, \tag{3}$$

assuming for simplicity that spatial coordinates commute,

$$\theta_{ij} = 0, \quad i, j\, \in \{1, 2, 3\} \tag{4}$$

and basing ourselves on the approach of Doplicher et al. [1]. Theory of classical waves and particles on such spacetimes was also formulated and applied to interference phenomena [8].

In this paper, we continue this line of investigation and study the following three algebras and their physics.

1) The Noncommutative Cylinder $\mathcal{A}_\theta\left(\mathbb{R} \times S^1\right)$

It is generated by $\hat{x}_0$ and $e^{-i\hat{x}_1}$ with the relation

$$\left[\,\hat{x}_0\,,\,e^{-i\hat{x}_1}\right] = \theta\,e^{-i\hat{x}_1}\,. \tag{5}$$

2) Noncommutative $\mathbb{R}^3$

The algebra $\hat{e}_2$ in this case is the enveloping algebra of e_2, the Lie algebra of the Euclidean group E_2. Spacetime coordinates $\hat{x}_\mu$ form a basis of e_2 and fulfill the commutation relations

$$[\,\hat{x}_i\,,\,\hat{x}_j] = 0\,, \quad [\,\hat{x}_0\,,\,\hat{x}_i] = i\theta\,\epsilon_{ij}\hat{x}_j\,,$$

$$\epsilon_{ij} = -\epsilon_{ji}\,, \quad \epsilon_{12} = 1\,. \tag{6}$$

Thus $\hat{x}_i$ are identified with translations and $\hat{x}_0/\theta$ is the canonically normalised angular momentum J:

$$e^{i2\pi J}\hat{x}_\mu e^{-i2\pi J} = \hat{x}_\mu\,. \tag{7}$$

The Lie algebra e_2 is a contraction of $so(2,1)$, the Lie algebra of $SO(2,1)$. The latter and its enveloping algebra have occurred as spacetime algebras in 2+1 gravity [3]. (See also [4]-[6] and also [9] in this connection.)

3) The Noncommutative $\mathbb{R}\times S^3$, $\mathcal{A}_\theta\left(\mathbb{R}\times S^3\right)$

We can represent $S^3 = \langle x\in\mathbb{R}^4\ :\ \sum_\lambda x_\lambda^2 = 1\rangle$ by $SU(2)$ matrices:

$$x_0\mathbb{I} + i\vec{\tau}\cdot\vec{x}\ \in SU(2)\,, \tag{8}$$

where $\mathbb{I}$ is the 2×2 unit matrix and τ_i are Pauli matrices. In this way we identify S^3 and $SU(2)$. Left- and right- regular representations of $SU(2)$ act on functions $\mathcal{C}^\infty$ $(SU(2))$ on $SU(2)$.

Let $su(2)$ be the Lie algebra of $SU(2)$ with conventional angular momentum operators J_i. Then in particular, J_3 has a right action J_3^R on $\mathcal{C}^\infty$ $(SU(2))$:

$$\left(e^{i\theta J_3^R}\,\hat{f}\right)(g) = \hat{f}\ \left(g\,e^{i\theta J_3/2}\right) \tag{9}$$

for $\hat{f}\in\mathcal{C}^\infty$ $(SU(2))$ and $g\in SU(2)$.

In the algebra $\mathcal{A}_\theta\ (\mathbb{R}\times S^3)$, the spatial slice S^3 is represented by the commutative algebra $\mathcal{C}^\infty\ (SU(2))$, and time $\hat{x}_0$ is identified with $2\theta J_3^R$ in the following way:

$$\left(e^{i\omega\hat{x}_0}\,\hat{f}\,e^{-i\omega\hat{x}_0}\right)(s) := \left(e^{i\omega 2\theta J_3^R}\,\hat{f}\right)(s)\,. \tag{10}$$

Cases 1) and 3) are actually very similar. In case 1), the spatial slice has algebra $C^\infty\,(S^1)$ and $J = \hat{x}_0/\theta$ is the canonically normalised generator of rotations: $e^{i2\pi J}\hat{\alpha}\,e^{-i2\pi J} = \hat{\alpha}$, for $\hat{\alpha}\in\mathcal{A}_\theta\,(\mathbb{R}\times S^1)$.

In all these cases, time translations get quantised in units of θ in quantum physics. This result is known for cases 1) and 2) [2]-[6]. It comes from the fact that the spectrum $\operatorname{spec} J$ or $\operatorname{spec} J_3$ of J or J_3 in an irreducible representation of the associated algebra is spaced in units of θ. We will prove it fully as we go along.

Using a different approach, a model with quantised evolution was also constructed in [9].

There are generalisations of these constructions to manifolds $\mathbb{R}\times M$ where $\mathbb{R}$ accounts for time and M is the spatial slice, provided M admits a $U(1)$ action. If J is its generator on $\mathcal{C}^\infty(M)$, we can set $\hat{x}_0 = \theta J$ and get an algebra $\mathcal{A}_\theta\,(\mathbb{R}\times M)$ with quantised evolution.

The mathematical approach to noncommutativity in this paper is similar to that of Rieffel, Connes, Landi and others [10]-[11]. We have drawn much inspiration from their work.

After reviewing [7] in the next section, we will study the three preceding examples in the subsequent sections. Issues related to energy nonconservation and also scattering and quantum field theory are taken up after that.

II. The Noncommutative Plane

1. Generalities

The noncommutative or Moyal plane $A_\theta\left(\mathbb{R}^d\right)$ is based on the commutation relation (2). We outline how to do quantum physics on this algebra, summarising [7]. It is enough to consider $d=2$. Then we can write, without loss of generality,

$$[\hat{x}_\mu\,,\,\hat{x}_\nu] = i\theta\epsilon_{\mu\nu}, \quad \mu,\nu\in\ \{0\,,\,1\},$$

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}\,, \quad \epsilon_{01} = -\epsilon_{10} = 1\,, \quad \theta\geq 0\,. \tag{11}$$

Since $\hat{x}_1$ and $-\hat{x}_0/\theta$ have the commutation relation of position and momentum, we can realise this algebra irreducibly on $L^2(\mathbb{R})$ as in elementary quantum mechanics.

For quantum mechanics on $\mathcal{A}_\theta(\mathbb{R}^2)$, elements of $\mathcal{A}_\theta(\mathbb{R}^2)$ itself constitute the "wave functions" or "vector states". Then for $\hat{\alpha} \in \mathcal{A}_\theta(\mathbb{R}^2)$, we have two linear operators $\hat{\alpha}^{L,R}$ on these vector states:

$$\hat{\alpha}^L \hat{\psi} = \hat{\alpha}\hat{\psi}\,, \quad \hat{\alpha}^R \hat{\psi} = \hat{\psi}\hat{\alpha}\,, \quad \hat{\psi} \in \mathcal{A}_\theta(\mathbb{R}^2)\,. \tag{12}$$

Their difference $ad\,\hat{\alpha}$ gives the adjoint action of $\hat{\alpha}$:

$$ad\,\hat{\alpha} = \hat{\alpha}^L - \hat{\alpha}^R, \quad ad\,\hat{\alpha}\,\hat{\psi} = [\hat{\alpha}\,,\,\hat{\psi}]\,. \tag{13}$$

For $\theta = 0$, we have time- and space- translation generators $i\partial_t$, $-i\frac{\partial}{\partial x_1}$. The latter is the momentum. Their analogues here are $\hat{P}_0$, $\hat{P}_1$ where

$$\hat{P}_0 = -\frac{1}{\theta} ad\,\hat{x}_1\,, \quad \hat{P}_1 = -\frac{1}{\theta} ad\,\hat{x}_0\,, \tag{14}$$

$$\hat{P}_\mu \hat{x}_\nu = i\eta_{\mu\nu}\mathbb{I}\,, \quad \eta_{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu\,, \quad \eta_{00} = -\eta_{11} = 1\,. \tag{15}$$

2. The Inner Product

The next step is to find an inner product on the vector states. There are several (equivalent) possibilities [7]. We describe one here.

Let

$$\hat{\psi} = \int d^2k\,\tilde{\psi}(k)\,e^{ik_1\hat{x}_1}\,e^{ik_0\hat{x}_0}\,. \tag{16}$$

We define its symbol ψ, which is a complex function on $\mathbb{R}^2 = \{x = (x_0\,,\,x_1)\}$, by

$$\psi = \int d^2k\,\tilde{\psi}(k)\,e^{ik_1 x_1}\,e^{ik_0 x_0}\,. \tag{17}$$

The inner product $(\hat{\psi}\,,\,\hat{\eta})_{x_0}$ of two vector states $\hat{\psi}$, $\hat{\eta}$ is then

$$(\hat{\psi}\,,\,\hat{\eta})_{x_0} = \int dx_1\,\psi^*(x_0\,,x_1)\,\eta(x_0\,,x_1)\,, \tag{18}$$

η being the symbol of $\hat{\eta}$. It depends on x_0.

3. The Hilbert Space

If ψ vanishes at "time" x_0 and all x_1, $\hat{\psi}$ is a null vector in this inner product. The set $\mathcal{N}$ of null vectors in this inner product is thus large. The inner product depends as well on x_0. Both the nontrivial null vectors and dependence on x_0 can be eliminated by imposing the Schrödinger equation or "constraint". The completion of the resultant vector states in the scalar product is the quantum Hilbert space $\mathcal{H}$.

Let $\hat{H}$ be a "time-independent" Hamiltonian hermitian in the inner product:

$$[\hat{P}_0\, ,\, \hat{H}] = 0, \quad (\hat{\psi}\, ,\, \hat{H}\hat{\eta})_{x_0} = (\hat{H}\hat{\psi}\, ,\, \hat{\eta})_{x_0}\, . \tag{19}$$

Let $\mathcal{C}$ be the vectors fulfilling the Schrödinger equation:

$$\mathcal{C} = \left\langle \hat{\psi} \in \mathcal{A}_\theta\left(\mathbb{R}^2\right) : \left(\hat{P}_0 - \hat{H}\right)\hat{\psi} = 0 \right\rangle\, . \tag{20}$$

Then since

$$\left(e^{-i\hat{P}_0\tau}\hat{\psi}\, ,\, e^{-i\hat{P}_0\tau}\hat{\eta}\right)_{x_0} = \left(e^{-i\hat{H}\tau}\hat{\psi}\, ,\, e^{-i\hat{H}\tau}\hat{\eta}\right)_{x_0} = \left(\hat{\psi}\, ,\, \hat{\eta}\right)_{x_0} \tag{21}$$

for $\hat{\psi}\, ,\, \hat{\eta} \in \mathcal{C}$, we can see that the inner product is independent of x_0 for vectors in $\mathcal{C}$. So we write $(\hat{\psi}\, ,\hat{\eta})_{x_0}$ as $(\hat{\psi}\, ,\hat{\eta})$.

Also if ψ_τ is the symbol of $e^{-i\hat{P}_0\tau}\hat{\psi}$, then (16), (17) show that $\psi_\tau(x_0\, ,\, x_1) = \psi(x_0 + \tau\, ,\, x_1)$. Hence if $\hat{\psi} \in \mathcal{C}$ and ψ is zero at x_0 and all x_1, then it is identically zero. From (17), $\tilde{\psi}$ is then zero, and so by (16), $\hat{\psi} = 0$. So we also get

$$\mathcal{N} \cap \mathcal{C} = \{0\} \tag{22}$$

as claimed.

There is a simple solution for the Schrödinger constraint for time-independent $\hat{H}$:

$$\hat{\psi} \in \mathcal{C} \Longrightarrow \hat{\psi} = e^{-i\hat{H}\hat{x}_0^R}\hat{\chi}\left(\hat{x}_1\right)\, . \tag{23}$$

The vector $\hat{\chi}\left(\hat{x}_1\right)$ has no dependence on $\hat{x}_0$, that is, commutes with $\hat{x}_1$.

We can extend the discussion to time-dependent, but still hermitean, Hamiltonians. For details, see [7].

4. On Positive Maps

A positive map S on a $*$-algebra $\mathcal{A}$ is a linear map $S : \hat{\alpha} \in \mathcal{A} \longmapsto S(\hat{\alpha}) \in \mathbb{C}$ with the properties

$$S(\hat{\alpha}^*) = S(\hat{\alpha})^* \,,$$

$$S\left(\hat{\alpha}^*\hat{\alpha}\right) \geq 0 \,. \tag{24}$$

Given such a map S, we can define an inner product $(\cdot\,,\,\cdot)$ on $\mathcal{A}$: $(\hat{\alpha}\,,\,\hat{\beta}) = S(\hat{\alpha}^*\hat{\beta})$.

The inner product (18) comes from the following positive map S_{x_0}:

$$S_{x_0}(\hat{\psi}) = \int dx_1\, \psi(x_0\,,\,x_1)\,. \tag{25}$$

Positive maps, like traces, can substitute for integration on algebras, and are useful for formulating physical theories on noncommutative spacetimes.

III. The Noncommutative Cylinder

The noncommutative cylinder $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$ has been considered in great detail by Chaichian et al. [2], especially as regards its quantum field theory aspects. They have pointed out and emphasised that time gets quantised on $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$ (see also [9]) and studied the impact of this quantisation on causality and unitarity. Below, we review how this quantisation comes about and develop quantum physics on $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$. We do not encounter problems with unitarity.

For $\theta = 0$, there is a close relation between $C^\infty\left(\mathbb{R}\times\mathbb{R}\right)$ and the functions $C^\infty\left(\mathbb{R}\times S^1\right)$ on a cylinder. The former is generated by coordinate functions $\hat{x}_0$ and $\hat{x}_1$, and the latter by $\hat{x}_0$ and $e^{i\hat{x}_1}$, $e^{i\hat{x}_1}$ being invariant under the 2π-shifts $\hat{x}_1 \to \hat{x}_1 \pm 2\pi$. Following this idea, we can regard the noncommutative $\mathbb{R}\times S^1$ algebra $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$ as generated by $\hat{x}_0$ and $e^{i\hat{x}_1}$ with the defining relation

$$e^{i\hat{x}_1}\hat{x}_0 = \hat{x}_0 e^{i\hat{x}_1} + \theta e^{i\hat{x}_1} \,,$$

following from (11).

For $C^\infty\left(\mathbb{R}\times S^1\right)$, the momentum $\hat{p}_1$ is the differential operator defined by

$$\left[\hat{p}_1, e^{i\hat{x}_1}\right] = e^{i\hat{x}_1} \,. \tag{26}$$

By evaluating (26) at x_1, we can write it in the usual way: $\left[-i\frac{\partial}{\partial x_1}, e^{ix_1}\right] = e^{ix_1}$.

It follows from (26) that

$$e^{i2\pi\hat{p}_1} e^{i\hat{x}_1} e^{-i2\pi\hat{p}_1} = e^{i\hat{x}_1} \,. \tag{27}$$

So $e^{i2\pi\hat{p}_1}$ is in the center of the algebra generated by $\hat{p}_1$, $e^{i\hat{x}_1}$ with the relation (26). In an irreducible representation (IRR), it is a phase $e^{i\varphi}$ times $\mathbb{I}$. The spectrum of $\hat{p}_1$ in an IRR is hence

$$\operatorname{spec}\hat{p}_1 = \mathbb{Z} + \frac{\varphi}{2\pi} \equiv \left\{n + \frac{\varphi}{2\pi} : n \in \mathbb{Z}\right\} \,. \tag{28}$$

Its domain $\mathcal{D}_\varphi(\hat{p}_1)$ in such an IRR is spanned by quasi-periodic functions χ_n:

$$\chi_n = e^{i\left(n+\frac{\varphi}{2\pi}\right)\hat{x}_1} \,, \quad n \in \mathbb{Z} \,,$$

$$\chi_n(\hat{x}_1 + 2\pi) = e^{i\varphi}\chi_n(\hat{x}_1) \,. \tag{29}$$

If for example

$$H = \frac{\hat{p}_1^2}{2m} \tag{30}$$

is the Hamiltonian, its domain $\mathcal{D}_\varphi(H)$ fulfilling the Schrödinger constraint as well is spanned by

$$\psi_n = \chi_n e^{-iE_n\hat{x}_0} \,, \tag{31}$$

with ψ_n being eigenstates of H:

$$H\psi_n = E_n\psi_n \,, \tag{32}$$

$$E_n = \frac{1}{2m}\left(n + \frac{\varphi}{2\pi}\right)^2 \,. \tag{33}$$

The quantity φ is generally interpreted as the flux through the circle.

For the noncommutative cylinder, (27) generalises in a striking manner:

$$e^{-i\frac{2\pi}{\theta}\hat{x}_0} e^{i\hat{x}_1} e^{i\frac{2\pi}{\theta}\hat{x}_0} = e^{i\hat{x}_1} \,. \tag{34}$$

Hence in an IRR,

$$e^{-i\frac{2\pi}{\theta}\hat{x}_0} = e^{-i\varphi}\mathbb{I}\,, \tag{35}$$

so that for the spectrum $\operatorname{spec}\hat{x}_0$ of $\hat{x}_0$ in an IRR, we have,

$$\operatorname{spec}\hat{x}_0 = \theta\mathbb{Z} + \frac{\theta\varphi}{2\pi} = \theta\left(\mathbb{Z} + \frac{\varphi}{2\pi}\right) \equiv \left\{\theta\left(n + \frac{\varphi}{2\pi}\right) : n \in \mathbb{Z}\right\}\,. \tag{36}$$

We can realise $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$ irreducibly in the auxiliary Hilbert space $L^2\left(S^1, dx_1\right)$. It has the scalar product given by

$$(\alpha,\beta) = \int_0^{2\pi} dx_1\, \alpha^*\left(e^{ix_1}\right)\beta\left(e^{ix_1}\right)\ ,\ \alpha,\beta \in L^2\left(S^1, dx_1\right)\,. \tag{37}$$

On this space, $e^{i\hat{x}_1}$ acts by evaluation map,

$$\left(e^{i\hat{x}_1}\alpha\right)\left(e^{ix_1}\right) = e^{ix_1}\alpha\left(e^{ix_1}\right)\,, \tag{38}$$

while $\hat{x}_0/\theta$ acts like the $\theta = 0$ momentum with domain $D_\varphi(\hat{p}_1)$.

We denote this particular representation of $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$ as $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$.

Let us examine $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$ more closely. We can regard it as generated by $e^{i\hat{x}_1}$ and $e^{i\omega\hat{x}_0}$ where ω is real. Now because of the spectral result (36),

$$e^{i\left(\omega+\frac{2\pi}{\theta}\right)\hat{x}_0} = e^{i\varphi}e^{i\omega\hat{x}_0}\,. \tag{39}$$

Thus elements of $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$ are quasiperiodic in ω just as χ_n is quasiperiodic in $\hat{x}_1$, and we can restrict ω to its fundamental domain:

$$\omega \in \left[-\frac{\pi}{\theta}, \frac{\pi}{\theta}\right]\,. \tag{40}$$

The general element of $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$ is thus

$$\hat{\alpha} = \sum_{n\in\mathbb{Z}} \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, \tilde{\alpha}_n(\omega) e^{in\hat{x}_1} e^{i\omega\hat{x}_0}\,, \tag{41}$$

as first discussed by Chaichian et al. [2].

1. Positive Maps and Inner Products

A positive map on $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$ can be found from symbol calculus. Since the spectrum of $\hat{x}_0$ is $\theta\left(\mathbb{Z} + \frac{\varphi}{2\pi}\right)$ and the spectrum of $e^{i\hat{x}_1}$ is S^1, the symbol of $\hat{\alpha}$ is a function α on $\theta\left(\mathbb{Z} + \frac{\varphi}{2\pi}\right) \times S^1$:

$$\alpha : \theta\left(\mathbb{Z} + \frac{\varphi}{2\pi}\right) \times S^1 \to \mathbb{C}\,. \tag{42}$$

It is defined by

$$\alpha\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right) = \sum_{n\in\mathbb{Z}} \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, \tilde{\alpha}_n(\omega) e^{inx_1} e^{i\omega\theta\left(m+\frac{\varphi}{2\pi}\right)} . \tag{43}$$

Before proceeding, we show that $\hat{\alpha}$ determines $\tilde{\alpha}_n$ and hence α uniquely, so that the map $\hat{\alpha} \to \alpha$ is well-defined. We show also the converse, that α determines $\tilde{\alpha}_n$ and hence $\hat{\alpha}$ uniquely, so that the map $\hat{\alpha} \to \alpha$ is bijective.

Let $|n\rangle$ be the normalised eigenstates of $\hat{x}_0$:

$$\hat{x}_0 \,|n\rangle = \theta\left(n+\frac{\varphi}{2\pi}\right)|n\rangle \ , \ \langle m|n\rangle = \delta_{mn} \ , \quad n \in \mathbb{Z} . \tag{44}$$

Then

$$e^{i\hat{x}_1}\,|n\rangle = |n-1\rangle \ . \tag{45}$$

Therefore

$$\langle m|\, \hat{\alpha}\, |n\rangle = \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, \tilde{\alpha}_{n-m}(\omega) e^{i\omega\theta\left(n+\frac{\varphi}{2\pi}\right)} , \tag{46}$$

and since

$$\frac{\theta}{2\pi}\sum_n e^{i(\omega-\omega')\theta n} = \delta(\omega-\omega') , \tag{47}$$

we find

$$\frac{\theta}{2\pi}\sum_n e^{-i\omega\theta\left(n+\frac{\varphi}{2\pi}\right)} \langle n-m|\, \hat{\alpha}\, |n\rangle = \tilde{\alpha}_m(\omega) . \tag{48}$$

The inverse map follows similarly:

$$\tilde{\alpha}_n(\omega) = \frac{\theta}{(2\pi)^2}\sum_m e^{-i\omega\theta\left(m+\frac{\varphi}{2\pi}\right)} \int_0^{2\pi} dx_1\, e^{-inx_1} \alpha\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right) . \tag{49}$$

Our positive map is $S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}$:

$$S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}(\hat{\alpha}) = \int_0^{2\pi} dx_1\, \alpha\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right) . \tag{50}$$

Just as in (18), we then have, for inner product,

$$\begin{aligned}\left(\hat{\alpha}, \hat{\beta}\right)_{\theta\left(m+\frac{\varphi}{2\pi}\right)} &= S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}\left(\hat{\alpha}^*\hat{\beta}\right) \\ &= \int_0^{2\pi} dx_1\, \alpha^*\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right) \beta\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right) .\end{aligned} \tag{51}$$

There are other possibilities for inner product such as the one based on coherent states. The equivalence of theories based on different inner products is discussed in [7].

2. Spectrum of Momentum

We can infer the spectrum of the momentum operator $\hat{P}_1$ when it acts on $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$. Since this algebra allows for only integral powers of $e^{i\hat{x}_1}$, and

$$\hat{P}_1 e^{in\hat{x}_1} = n e^{in\hat{x}_1}, \tag{52}$$

we have

$$\operatorname{spec}\hat{P}_1 = \mathbb{Z}\,. \tag{53}$$

The flux term is 0 in this spectrum.

For the construction of a Hilbert space, we do not need this algebra. It is enough to have an $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$-module which can be consistently treated. Such a module is

$$\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right) = \left\langle \hat{\gamma} = e^{i\frac{\psi}{2\pi}\hat{x}_1}\sum_{n\in\mathbb{Z}}\int_{-\frac{\pi}{\theta}}^{\frac{\pi}{\theta}} d\omega\tilde{\gamma}_n(\omega)e^{in\hat{x}_1}e^{i\omega\hat{x}_0}\right\rangle . \tag{54}$$

The eigenvalues of $\hat{P}_1$ are now shifted by $\frac{\psi}{2\pi}$:

$$\hat{P}_1 e^{i\frac{\psi}{2\pi}\hat{x}_1}e^{in\hat{x}_1} = \left(n+\frac{\psi}{2\pi}\right)e^{i\frac{\psi}{2\pi}\hat{x}_1}e^{in\hat{x}_1}\,,\; n\in\mathbb{Z}\,. \tag{55}$$

So we now have a flux term $\frac{\psi}{2\pi}$.

We have to check that $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$ also has an inner product. That is so because if

$$\hat{\gamma},\hat{\delta}\in\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)\,, \tag{56}$$

then

$$\hat{\gamma}^*\hat{\delta}\in\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right), \tag{57}$$

(the ψ-dependent factors $e^{i\frac{\psi}{2\pi}\hat{x}_1}$ cancelling out), so that the inner product is still like (51):

$$(\hat{\gamma},\hat{\delta})_{\theta\left(m+\frac{\varphi}{2\pi}\right)} = S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}(\hat{\gamma}^*\hat{\delta})\,. \tag{58}$$

It is interesting that the flux terms in time and momentum can be different.

We remark that the Schrödinger constraint below does not alter the spectrum of $\hat{P}_1$.

3. The Schrödinger Constraint

a) The Time-Independent Hamiltonian

Since

$$i\partial_{x_0} e^{i\omega\hat{x}_0} = -\omega e^{i\omega\hat{x}_0} \tag{59}$$

is not quasiperiodic in ω, continuous time translations and the Schrödinger constraint in the original form cannot be defined on $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$.

But translation of $\hat{x}_0$ by $\pm\theta$ leaves its spectrum intact. Hence the operator

$$e^{-i\theta\left(i\partial_{x_0}\right)} = e^{iad\,\hat{x}_1}\,, \tag{60}$$

and its integral powers act on $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$. The conventional Schrödinger constraint is thus changed to a discrete Schrödinger constraint. In the time-independent case when the Hamiltonian can be written as $\hat{H}\left(e^{i\hat{x}_1^L}, \hat{P}_1\right)$, the family of vector states constrained by the discrete Schrödinger equation is

$$\tilde{\mathcal{H}}_\theta\left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right) = \left\{\hat{\psi}\in\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right) : e^{-i\theta\left(i\partial_{x_0}\right)}\hat{\psi} = e^{-i\theta\hat{H}}\hat{\psi}\right\}. \tag{61}$$

It has solutions

$$\hat{\psi} = e^{-i\hat{x}_0^R\hat{H}\left(e^{i\hat{x}_1^L}, \hat{P}_1\right)} e^{i\frac{\psi}{2\pi}\hat{x}_1}\hat{\chi}\left(e^{i\hat{x}_1}\right)\,, \tag{62}$$

just as in (23).

b) The Time-Dependent Hamiltonian

The time-dependent Hamiltonian is

$$\hat{\hat{H}}\left(\hat{x}_0^L, \hat{x}_0^R, e^{i\hat{x}_1^L}, \hat{P}_1\right) \tag{63}$$

and the Schrödinger constraint (61) defining $\tilde{\mathcal{H}}_\theta\left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$ is intact. We can solve this constraint as follows.

Write

$$\hat{\hat{H}}\left(\hat{x}_0^L, \hat{x}_0^R, e^{i\hat{x}_1^L}, \hat{P}_1\right) = \hat{\hat{H}}\left(-\theta\hat{P}_1 + \hat{x}_0^R, \hat{x}_0^R, e^{i\hat{x}_1^L}, \hat{P}_1\right) \equiv \hat{H}\left(\hat{x}_0^R, e^{i\hat{x}_1^L}, \hat{P}_1\right). \tag{64}$$

We can try to replace $\hat{x}_0^R$ by τ and try to solve (61) along the lines of the treatment in [7] of time-dependent Hamiltonians for $\mathcal{A}_\theta(\mathbb{R}^2)$. But for that we need to know the spectrum $\operatorname{spec}\hat{x}_0^R$ of $\hat{x}_0^R$ in $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$, since $\hat{H}$ is defined in τ only for $\tau \in \operatorname{spec}\hat{x}_0^R$.

In $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$, we choose the following domain $D_\varphi\left(\hat{x}_0^R\right)$ for $\hat{x}_0^R$:

$$D_\varphi\left(\hat{x}_0^R\right) = \left\{\hat{\alpha} \in \mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right) : \tilde{\alpha}_n\left(\omega + \frac{2\pi}{\theta}\right) = e^{-i\varphi}\tilde{\alpha}_n\left(\omega\right)\right\}. \tag{65}$$

For $\hat{\alpha} \in D_\varphi\left(\hat{x}_0^R\right)$,

$$\hat{x}_0^R\hat{\alpha} = \hat{\alpha}\hat{x}_0 = e^{i\frac{\psi}{2\pi}\hat{x}_1}\sum_{n\in\mathbb{Z}}\int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\,\tilde{\alpha}_n(\omega)e^{in\hat{x}_1}\left(-i\frac{\partial}{\partial\omega}e^{i\omega\hat{x}_0}\right)$$

$$= -ie^{i\frac{\psi}{2\pi}\hat{x}_1}\sum_{n\in\mathbb{Z}} e^{in\hat{x}_1}\left[\tilde{\alpha}_n(\omega)e^{i\omega\hat{x}_0}\right]_{-\frac{\pi}{\theta}}^{\frac{\pi}{\theta}} + e^{i\frac{\psi}{2\pi}\hat{x}_1}\sum_{n\in\mathbb{Z}}\int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\left(i\frac{\partial}{\partial\omega}\tilde{\alpha}_n(\omega)\right)e^{in\hat{x}_1}e^{i\omega\hat{x}_0}. \tag{66}$$

The first (surface) terms vanish by (65) and

$$\hat{x}_0^R\hat{\alpha} = e^{i\frac{\psi}{2\pi}\hat{x}_1}\sum_{n\in\mathbb{Z}}\int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\,\left(i\frac{\partial}{\partial\omega}\tilde{\alpha}_n(\omega)\right)e^{in\hat{x}_1}e^{i\omega\hat{x}_0}. \tag{67}$$

Hence if

$$\hat{\alpha} = e^{i\frac{\psi}{2\pi}\hat{x}_1}\int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, e^{-i\omega\theta\left(n+\frac{\varphi}{2\pi}\right)}e^{in\hat{x}_1}e^{i\omega\hat{x}_0}, \tag{68}$$

then $\hat{\alpha}$ is an eigenvector of $\hat{x}_0^R$:

$$\hat{x}_0^R\hat{\alpha} = \theta\left(n + \frac{\varphi}{2\pi}\right)\hat{\alpha}, \tag{69}$$

and for the spectrum of $\hat{x}_0^R$, we get

$$\operatorname{spec}\hat{x}_0^R = \theta\mathbb{Z} + \frac{\theta\varphi}{2\pi}. \tag{70}$$

It is spaced in units of θ.

We can now solve (61):

$$\hat{\psi} = U\left(\hat{x}_0^R, -\infty\right)e^{i\frac{\psi}{2\pi}\hat{x}_1}\hat{\chi}\left(e^{i\hat{x}_1}\right), \tag{71}$$

where

$$U\left(\hat{x}_0^R, -\infty\right) = U\left(\hat{x}_0^R - \theta\right) U\left(\hat{x}_0^R - 2\theta\right) \dots ,$$

$$U\left(\hat{x}_0^R - j\theta\right) = e^{-i\left(\hat{x}_0^R - j\theta\right)\hat{H}\left(\hat{x}_0^R - j\theta,\, e^{i\hat{x}_1^L},\, \hat{P}_1\right)} . \tag{72}$$

Observables compatible with the Schrödinger constraint can be constructed as before.

4. Remarks

We point out that we can see the absence of nontrivial null states in $\tilde{\mathcal{H}}_\theta\left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$ as before so that the inner product becomes a true scalar product for $\tilde{\mathcal{H}}_\theta\left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$. Also, the Hilbert space $\mathcal{H}_\theta\left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$ obtained by completion of $\tilde{\mathcal{H}}_\theta\left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$ is independent of m in (58) while $\hat{x}_0^{L,R}$ do not act on $\mathcal{H}_\theta\left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$.

Note that while $e^{-i\frac{2\pi}{\theta}\hat{x}_0^R}$ acts on $\mathcal{H}_\theta\left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right)$, it is $e^{-i\varphi}\mathbb{I}$ because of (69). So it cannot be the starting point to define a time operator.

These remarks generalise to the other examples of discrete evolution considered below.

IV. Noncommutative $\mathbb{R}^3$

Here we show that the algebra $\hat{e}_2$ admits a positive map. With that, one can proceed to develop quantum physics.

If $\hat{x}_0$, $\hat{x}_a$ $(a = 1, 2)$ are time and space coordinate functions in commutative spacetime, we call their noncommutative analogues also by $\hat{x}_0$, $\hat{x}_a$. They fulfill the relations

$$[\hat{x}_a, \hat{x}_b] = 0 \ , \ \ a, b = 1, 2 \, ,$$

$$[\hat{x}_0, \hat{x}_a] = i\theta\varepsilon_{ab}\hat{x}_b \ , \ \varepsilon_{12} = -\varepsilon_{21} = 1 \ , \ \theta > 0 \, . \tag{73}$$

(73) defines the Lie algebra of the two-dimensional Euclidean group, and admits a $*$-operation: $\hat{x}_\mu^* = \hat{x}_\mu$. Equally important, it admits the time-translation automorphism $U(\tau) : U(\tau)\hat{x}_0 = \hat{x}_0 + \tau$. But it is not an inner automorphism, $\hat{x}_0$ having no conjugate operator.

Spatial translations are not automorphisms of (73). That means that momenta, free Hamiltonian or plane waves do not exist for (73).

The algebra $\hat{e}_2$ with relations (73) has been treated in detail by Chaichian et al. [2]. As they observe, the operator

$$\rho^2 = \sum \hat{x}_a \hat{x}_a \tag{74}$$

is in the center of $\hat{e}_2$. We can fix its value to be r^2 in an IRR just as we fixed the value of $e^{-i\frac{2\pi}{\theta}\hat{x}_0} \in \mathcal{A}_\theta\left(\mathbb{R} \times S^1\right)$. For $r^2 > 0$, we have the polar decomposition

$$\hat{x}_1 \pm i\hat{x}_2 = r e^{\mp i\hat{x}} \,. \tag{75}$$

Now

$$e^{i\hat{x}}\hat{x}_0 = \hat{x}_0 e^{i\hat{x}} + \theta e^{i\hat{x}} \,, \tag{76}$$

and $\hat{x}_0$, $e^{i\hat{x}}$ generate $\mathcal{A}_\theta\left(\mathbb{R} \times S^1\right)$, the algebra treated before. Hence we can borrow ideas from the treatment of $\mathcal{A}_\theta\left(\mathbb{R} \times S^1\right)$.

We briefly treat (73) regarding $\hat{x}_a$ as generators of $C^\infty\left(\mathbb{R}^2\right)$ and $\hat{x}_0/\theta$ as the generator of rotations in the $1-2$ plane. The algebra will be realised by operators on the auxiliary Hilbert space $L^2\left(\mathbb{R}^2, d^2x\right)$ with its standard scalar product $(\,.\,,\,.\,)$ where

$$(\alpha, \beta) = \int d^2x\, \alpha^*(x)\beta(x) \,. \tag{77}$$

On this space, $\hat{x}_a$ acts by evaluation map,

$$\hat{x}_a \alpha(x) = x_a \alpha(x) \,, \tag{78}$$

while $\hat{x}_0/\theta$ acts like angular momentum with

$$e^{i2\pi\hat{x}_0/\theta} = \mathbb{I} \,. \tag{79}$$

Then for the spectrum of $\hat{x}_0$,

$$\operatorname{spec} \hat{x}_0 = \theta\mathbb{Z} \,. \tag{80}$$

Time is quantised in units of θ as for $\mathcal{A}_\theta\left(\mathbb{R} \times S^1\right)$, but there is no shift from $\theta\mathbb{Z}$ by a flux term $\theta\varphi/2\pi$.

There are also ray representations of the Euclidean group which are representations of (73), where the spectrum $\theta\mathbb{Z}$ is shifted by a flux term $\frac{\theta\varphi}{2\pi}$. Our discussion can be adapted to this case as well.

We now give the positive map and inner product for $\hat{e}_2$.

The algebra $\hat{e}_2$ is generated by

$$e^{i\omega\hat{x}_0}\,, \quad e^{i\vec{p}.\hat{x}}\,, \quad \vec{p}.\hat{x} = p_1\hat{x}_1 + p_2\hat{x}_2\,, \quad \omega, p_a \in \mathbb{R}\,, \tag{81}$$

where because of the spectral condition (80),

$$e^{i\left(\omega+\frac{2\pi}{\theta}\right)\hat{x}_0} = e^{i\omega\hat{x}_0}\,. \tag{82}$$

Thus we restrict ω according to

$$|\omega| \leq \frac{\pi}{\theta}\,. \tag{83}$$

The general element of the algebra is

$$\hat{\alpha} = \int d^2p \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, \tilde{\alpha}(\omega,\vec{p}) e^{i\vec{p}.\hat{x}} e^{i\omega\hat{x}_0}\,. \tag{84}$$

The symbol we associate to $\hat{\alpha}$ is the function

$$\alpha : \theta\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{C}\,,$$

$$\alpha(\theta n, x) = \int d^2p \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, \tilde{\alpha}(\omega,\vec{p})\, e^{i\vec{p}.\vec{x}}\, e^{i\omega\theta n} \;,\; n \in \mathbb{Z}\,. \tag{85}$$

This gives the map

$$S_{\theta n}\left(\hat{\alpha}\right) = \int d^2x\, \alpha(\theta n, x)\,. \tag{86}$$

We can show that (86) is a positive map. We have the identity

$$e^{-i\omega\hat{x}_0}\hat{x}_a e^{i\omega\hat{x}_0} = R_{ab}\left(\theta\omega\right)\hat{x}_b \;,\; R\left(\theta\omega\right) = \begin{pmatrix} \cos\left(\theta\omega\right) & \sin\left(\theta\omega\right) \\ -\sin\left(\theta\omega\right) & \cos\left(\theta\omega\right) \end{pmatrix}\,. \tag{87}$$

A short calculation which uses this identity shows, in an obvious manner, that

$$S_{\theta n}\left(\hat{\alpha}^*\hat{\alpha}\right) = (2\pi)^2 \int d^2p \left| \int d\omega\, \tilde{\alpha}\left(\omega,\vec{p}\right) e^{i\omega\theta n} \right|^2 \geq 0\,. \tag{88}$$

Thus an inner product for $\hat{e}_2$ is

$$\left(\hat{\beta},\hat{\alpha}\right) = S_{\theta n}\left(\hat{\beta}^*\hat{\alpha}\right)\,. \tag{89}$$

V. The Noncommutative $\mathbb{R} \times S^3$

The noncommutative $\mathbb{R} \times S^3 \simeq \mathbb{R} \times SU(2)$ is denoted by $\mathcal{A}_\theta\left(\mathbb{R} \times S^3\right)$. Section I indicates its construction: we set the time operator $\hat{x}_0$ equal to $2\theta J_3^R$ where θ is the noncommutativity parameter. With $\mathcal{C}^\infty\left(SU(2)\right)$ denoting the commutative algebra of functions on $SU(2)$, $\mathcal{A}_\theta\left(\mathbb{R} \times S^3\right)$ is generated by $\mathcal{C}^\infty\left(SU(2)\right)$ and $\hat{x}_0$ with relation (10).

Let $L^2\left(SU(2)\,,\, d\mu\right)$ denote the Hilbert space of functions on $SU(2)$ with scalar product $(\cdot\,,\,\cdot)$ given by the Haar measure $d\mu$:

$$(\hat{a}\,,\,\hat{b}) = \int d\mu(s)\,\hat{a}^*(s)\,\hat{b}(s)\,. \tag{90}$$

Then $\mathcal{A}_\theta\left(\mathbb{R} \times S^3\right)$ acts naturally on this Hilbert space, $\mathcal{C}^\infty(SU(2))$ acting by point-wise multiplication and $\hat{x}_0$ as the differential operator $2\theta J_3^R$.

The spectrum spec J_3^R of J_3^R is $\mathbb{Z}/2$. Hence $\operatorname{spec}\hat{x}_0 = \theta\mathbb{Z}$. Therefore

$$e^{i2\pi\hat{x}_0/\theta} = \mathbb{I}\,. \tag{91}$$

It follows that time evolution is quantised in units of θ.

Furthermore

$$e^{i\left(\omega+\frac{2\pi}{\theta}\right)\hat{x}_0} = e^{i\omega\hat{x}_0}\,. \tag{92}$$

Hence we can restrict ω to $\left[-\frac{\pi}{\theta}\,,\,\frac{\pi}{\theta}\right]$ and represent an element $\hat{\psi}$ of $\mathcal{A}_\theta\left(\mathbb{R} \times S^3\right)$ as

$$\hat{\psi} = \int_{-\pi/\theta}^{\pi/\theta} d\omega\,\hat{\psi}_\omega\,e^{i\omega\hat{x}_0}, \quad \hat{\psi}_\omega \in \mathcal{C}^\infty(SU(2))\,. \tag{93}$$

The symbol of $\hat{\psi}$ is the function $\psi : (\operatorname{spec}\hat{x}_0 = \theta\mathbb{Z}) \times SU(2) \longrightarrow \mathbb{C}$ defined by

$$\psi(\theta n\,,\,s) = \int_{-\frac{\pi}{\theta}}^{\frac{\pi}{\theta}} d\omega\,\hat{\psi}_\omega(s)\,e^{i\omega\theta n}\,, \quad n \in \mathbb{Z}\,. \tag{94}$$

The inner product can be obtained from an associated map $S_{\theta n}$:

$$S_{\theta n}(\hat{\psi}) = \int d\mu(s)\,\psi(\theta n\,,s)\,. \tag{95}$$

We can check using the right-invariance of the Haar measure that

$$S_{\theta n}(\hat{\psi}^* \hat{\varphi}) = \int d\mu(s)\, \psi^*(\theta n, s)\, \varphi(\theta n, s)\,, \tag{96}$$

where φ is the symbol of $\hat{\varphi}$. Hence $S_{\theta n}$ is a positive map.

The rest of the treatment involving the Schrödinger constraint follows previous sections.

VI. On Energy Conservation

We focus on time-independent Hamiltonians $\hat{H}$. In that case, the Schrödinger constraint such as (61) is preserved by $\hat{H}$,

$$\hat{\psi} \in \tilde{\mathcal{H}}_\theta \left(e^{i\frac{\varphi}{2\pi}}\,,\, e^{i\frac{\psi}{2\pi}} \right) \Longrightarrow \hat{H}\hat{\psi} \in \tilde{\mathcal{H}}_\theta \left(e^{i\frac{\varphi}{2\pi}}\,,\, e^{i\frac{\psi}{2\pi}} \right)\,, \tag{97}$$

and consequently $\hat{H}$ is an observable for $\mathcal{A}_\theta\,(\mathbb{R} \times S^1)$. The same is true for $\hat{e}_2$ and $\mathcal{A}_\theta\,(\mathbb{R} \times S^3)$.

However time evolution involves

$$U(\theta) = e^{-i\theta \hat{H}}\,, \tag{98}$$

its inverse and powers. It is the same for $\hat{H}$ and $\hat{H} + \frac{2\pi}{\theta}$. Hence time evolution need conserve energy only mod $\frac{2\pi}{\theta}$.

This energy nonconservation should show up in scattering and decay processes. In either case, if E_i and E_f are initial and final energies, then for $\theta = 0$, energy conservation is enforced by the factor

$$\int_{-\infty}^{\infty} d\tau\, e^{-i\tau(E_f - E_i)} = 2\pi\delta(E_f - E_i) \tag{99}$$

in the scattering matrix element. For quantised evolutions such as ours, the factor becomes

$$\sum_{n\in\mathbb{Z}} e^{-in\theta(E_f - E_i)} = 2\pi\delta_{S^1}\left[\theta(E_f - E_i)\right] \tag{100}$$

where δ_{S^1} is the δ-function on S^1: $\delta_{S^1}(\theta + 2\pi) = \delta_{S^1}(\theta)$. Thus from an initial state of energy E_i, there can be transitions to energies $E_f = E_i + \frac{2\pi}{\theta}n,\ n \in \mathbb{Z}$.

In specific models, the probability $P_n(E)$ for transitions from $E_i = E$ to $E_f = E + \frac{2\pi}{\theta}n$ can be calculated. We initiate the theory for this purpose in the next section. We are looking for a manageable model for a specific calculation.

Suppose that we start with a state of sharp energy E and let it undergo multiple scattering. Let the probability for finding energy $E + \frac{2\pi}{\theta}n$ after k scatterings be $P_n(E\,,\,k)$. Then

$$P_n(E\,,\,k+1) = \sum_m P_{n-m}\left(E + \frac{2\pi}{\theta}m\,,\,1\right) P_m(E\,,\,k) \tag{101}$$

where

$$P_n(E\,,\,1) = P_n(E)\,. \tag{102}$$

Equation (101) defines a Markov process with $P_n(E\,,\,1)$ giving the rule for updating at each step. It is of considerable interest to study $P_n(E\,,\,k)$ and its limit $k \to \infty$.

We remark that the limiting distribution $P_n(E\,,\,\infty)$ may be of use to provide bounds on θ in conjunction with cosmological data. Presumably distant star or quasar signals arrive at us after a large number of scattering processes. We can imagine estimating their frequency dispersion after accounting for energy loss by standard $\theta = 0$ effects, and getting information on θ therefrom.

VII. Scattering Theory

We consider only a situation where the Hamiltonian $\hat{H}$ is time-independent.

The transition amplitude from the in state vector $|+\,,\,\alpha\rangle$ with label α to an out state vector $|-\,,\,\beta\rangle$ with label β defines the matrix element $\mathcal{S}_{\beta\alpha}$ of the S-matrix $\mathcal{S}$:

$$\mathcal{S}_{\beta\alpha} = \langle -\,,\,\beta|+\,,\,\alpha\rangle\,. \tag{103}$$

Let $\hat{H}_0$ be the "free" or "comparison" Hamiltonian. Then $|+\,,\,\alpha\rangle$ has the property

$$U(\theta)^N|+\,,\,\alpha\rangle = U_0(\theta)^N|\alpha\rangle \quad \text{as } N \to -\infty, \quad \text{with } N \in \mathbb{Z} \tag{104}$$

where

$$U_0(\theta) = e^{-i\theta \hat{H}_0}\,. \tag{105}$$

The meaning of (104) is that in the distant past, $|+\,,\,\alpha\rangle$ evolves like the free evolution of the vector $|\alpha\rangle$.

The label α can be given a meaning in terms of observables of the free system such as energy.

The limit involved requires care. It is to be understood in the strong sense. It defines the Møller operator

$$\Omega^+ = \lim_{\substack{N\to-\infty\,,\\ N\in\mathbb{Z}}} U(\theta)^{-N} U_0(\theta)^N \tag{106}$$

with the properties

$$\Omega^+|\alpha\rangle = |+\,,\,\alpha\rangle\,, \tag{107}$$

$$\Omega^+ e^{-i\theta\hat{H}_0} = e^{-i\theta\hat{H}}\Omega^+\,. \tag{108}$$

Equation (107) follows from (104) while the proof of (108) is as follows:

$$\Omega^+ e^{-i\theta\hat{H}_0} = \lim_{\substack{N\to-\infty\,,\\ N\in\mathbb{Z}}} U(\theta)^{-N} U_0(\theta)^{N+1} = \lim_{\substack{N'\to-\infty\,,\\ N'\in\mathbb{Z}}} U(\theta)^{-(N'-1)} U_0(\theta)^{N'} = e^{-i\theta\hat{H}}\Omega^+\,. \tag{109}$$

Thus Ω^+ intertwines the quantised evolutions due to $\hat{H}_0$ and $\hat{H}$.

For $\theta = 0$, time t is continuous. In that case, (108) is replaced by

$$\Omega^+ e^{-it\hat{H}_0} = e^{-it\hat{H}}\Omega^+\,. \tag{110}$$

So for $\theta = 0$, by differentiating in t, we get the stronger result

$$\Omega^+ \hat{H}_0 = \hat{H}\Omega^+\,. \tag{111}$$

But we cannot get such a stronger equation from (108) for $\theta \neq 0$. This is yet another indication that for $\theta \neq 0$, energy is conserved only mod $\frac{2\pi}{\theta}$.

Just as $|+\,,\,\alpha\rangle$ fulfills the Schrödinger constraint involving $\hat{H}$, $|\alpha\rangle$ fulfills the Schrödinger constraint involving $\hat{H}_0$ as follows from (108):

$$e^{-i\theta\hat{P}_0}|\alpha\rangle = e^{-i\theta\hat{H}_0}|\alpha\rangle \,. \tag{112}$$

So scalar products involving $|\alpha\rangle$'s are also time-independent and admit a general solution of a form such as (62).

In a similar way, if

$$\Omega^- = \lim_{\substack{M\to\infty\,,\\ M\in\mathbb{Z}}} U(\theta)^{-M}U_0(\theta)^M \,, \tag{113}$$

then

$$\Omega^-|\beta\rangle = |-\,,\,\beta\rangle \,, \tag{114}$$

$$\Omega^- e^{-i\theta\hat{H}_0} = e^{-i\theta\hat{H}}\Omega^- \,. \tag{115}$$

Hence

$$\mathcal{S}_{\beta\alpha} = \lim_{\substack{M\to\infty\,,\\ N\to-\infty\,,\\ M,N\in\mathbb{Z}}} \langle\beta|U_0(\theta)^{-M}U(\theta)^{M-N}U_0(\theta)^N|\alpha\rangle := \lim_{\substack{M\to\infty\,,\\ N\to-\infty\,,\\ M,N\in\mathbb{Z}}} \langle\beta|U_I(\theta,M,N)|\alpha\rangle \,, \tag{116}$$

$$U_I(\theta,M,N) = U_0(\theta)^{-M}U(\theta)^{M-N}U_0(\theta)^N = e^{iM\theta\hat{H}_0}e^{-i(M-N)\theta\hat{H}}e^{-iN\theta\hat{H}_0} \,. \tag{117}$$

In commutative physics, where $\theta = 0$, the corresponding expression $U_I(t\,,\,t')$ is

$$U_I(t\,,t') = e^{it\hat{H}_0}e^{-i(t-t')\hat{H}}e^{-it'\hat{H}_0} = T\,exp\left\{-i\int_{t'}^{t} d\tau\,\hat{H}_I(\tau)\right\} \,, \tag{118}$$

$$\hat{H}_I(\tau) = e^{i\hat{H}_0\tau}(\hat{H}-\hat{H}_0)e^{-i\hat{H}_0\tau} \,, \tag{119}$$

T denoting time-ordering, the interaction representation S-matrix being $U_I(\infty\,,\,-\infty)$.

Comparison of (117) and (118) shows that

$$U_I(\theta,M,N) = T\,exp\left\{-i\int_{N\theta}^{M\theta} d\tau\,\hat{H}_I(\tau)\right\} \,, \tag{120}$$

$$\hat{H}_I(\tau) = e^{i\hat{H}_0\tau}(\hat{H} - \hat{H}_0)e^{-i\hat{H}_0\tau}. \tag{121}$$

For $\theta = 0$, (118) has a power series expansion in $\hat{H}_I$. But there is a problem with such an expansion of (117): $U(\theta)$, $U_0(\theta)$ and $U_I(\theta, M, N)$ are invariant under *separate* shifts of $\hat{H}$ and $\hat{H}_0$ by $\pm\frac{2\pi}{\theta}$, however $\hat{H}_I(\tau)$ and hence the terms of the perturbation series are invariant only under the *joint* shift of both by the same amount, the joint shift leaving $\hat{H}_I(\tau)$ invariant. Thus perturbative approximation disturbs an essential feature of quantised evolution.

It remains to find a substitute for perturbation theory. Perhaps an approximation based on the K-matrix formalism and effective range expansion [12], [13] may be acceptable.

VIII. On Quantum Fields

As the spacetime algebras of our interest admit only quantised time evolutions as automorphisms, a field cannot be the solution of a Klein-Gordon or Dirac equation. We need another approach to quantising spacetime fields for purposes of constructing quantum fields.

One way is to define the quantum field $\hat{\Phi}$ by expanding it in a basis of orthonormal solutions of the Schrödinger constraint. The coefficients of the expansion would be annihilation operators. This is a common approach in condensed matter theory.

For specificity consider $\mathcal{A}_\theta\left(\mathbb{R} \times S^1\right)$ and the "free" Hamiltonian

$$\hat{H}_0 = \frac{\hat{P}_1^2}{2M}\,. \tag{122}$$

In that case, $\tilde{\mathcal{H}}_\theta\left(e^{i\frac{\varphi}{2\pi}}\,,\, e^{i\frac{\psi}{2\pi}}\right)$ of (61) is spanned by

$$\hat{\psi}_n = \frac{1}{\sqrt{2\pi}}e^{i\left(n+\frac{\psi}{2\pi}\right)\hat{x}_1}e^{-i\omega_n\hat{x}_0}\,, \tag{123}$$

where

$$\omega_n = \frac{1}{2M}\left(n + \frac{\psi}{2\pi}\right)^2\,, \tag{124}$$

$$\hat{H}_0\hat{\psi}_n = \frac{1}{2M}\left(n + \frac{\psi}{2\pi}\right)^2\hat{\psi}_n\,, \tag{125}$$

$$(\hat{\psi}_m\,,\,\hat{\psi}_n)_{\theta\left(m+\frac{\varphi}{2\pi}\right)}=\delta_{mn}\,. \tag{126}$$

We can now write

$$\hat{\Phi}=\sum_n a_n\hat{\psi}_n,\quad \left[a_n\,,\,a_m^\dagger\right]=\delta_{nm} \tag{127}$$

where $\hat{\Phi}$ describes a free "nonrelativistic" spin-zero field. (We will not consider higher spins in this sketch.)

The second-quantised free Hamiltonian associated with $\hat{\Phi}$ is

$$\hat{\hat{H}}_0=\sum_n \omega_n a_n^\dagger a_n\,. \tag{128}$$

$\hat{\Phi}$ fulfills the second-quantised Schrödinger constraint:

$$e^{-i\theta\hat{P}_0}\hat{\Phi}=U_0(\theta)^{-1}\hat{\Phi}U_0(\theta)\,, \tag{129}$$

$$U_0(\theta)\equiv e^{-i\theta\hat{\hat{H}}_0}\,. \tag{130}$$

The next step is to introduce an interaction Hamiltonian. We follow earlier works [1], [7] in this regard. An example of an interaction Hamiltonian in interaction representation is

$$\hat{\hat{H}}_I(\tau)=:\,e^{i\tau\hat{\hat{H}}_0}\lambda S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}\left(\hat{\Phi}^\dagger\hat{\Phi}\,\hat{\Phi}^\dagger\hat{\Phi}\right)e^{-i\tau\hat{\hat{H}}_0}\;: \tag{131}$$

where $:\,\cdot\,:$ denotes normal ordering of a_n, $a_n^\dagger$.

$U_I(\theta,M,N)$ follows from (120):

$$U_I(\theta,M,N)=T\,exp\left\{-i\int_{N\theta}^{M\theta}d\tau\,\hat{\hat{H}}_I(\tau)\right\}\;,\;M,N\in\mathbb{Z}\,, \tag{132}$$

the S-matrix being

$$\mathcal{S}=\lim_{\substack{M\to\infty\,,\\ N\to-\infty\,,\\ M,N\in\mathbb{Z}}}U_I(\theta,M,N)\,. \tag{133}$$

As before, perturbation series, term by term, is not invariant under the shifts of $\hat{\hat{H}}_I(\tau)$ by $\pm\frac{2\pi}{\theta}$, whereas (132) is. That leaves us with a problem.

It is also important to know if and how $\mathcal{S}$ depends on $\left(m+\frac{\varphi}{2\pi}\right)$.

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